Quasi-exact solvability and intertwining relations

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Abstract

We emphasize intertwining relations as a universal tool in constructing one-dimensional quasi-exactly solvable operators and offer their possible generalization to the multidimensional case. Considered examples include all quasi-exactly solvable operators with invariant subspaces of monomials. We show that the simplest case of generalized intertwining relations allows to naturally relate quasi-exactly solvable operators with two invariant monomial subspaces to a nonlinear parasupersymmetry of second order. Quantum-mechanical systems with linear and nonlinear supersymmetry are discussed from the viewpoint of quasi-exact solvability. We construct such a general system with a cubic supersymmetry and argue that quantum-mechanical systems with nonlinear supersymmetry of fourth order and higher are generally not quasi-exactly solvable. Besides, we construct two examples of quasi-exactly solvable operators with invariant subspaces which cannot be reduced to monomial spaces.

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1 Introduction

In the eighties a new class of quantum-mechanical problems was discovered. Spectra of such systems can be partially found by an algebraic procedure, therefore they were called *quasi-exactly solvable* systems. In this sense these problems occupies an intermediate position between exactly solvable and unsolvable systems. There were elaborated several approaches to constructing quasi-exactly solvable quantal systems. Among them the Lie-algebraic construction [1] and analytic approach [2] are most common. In the Lie-algebraic approach a quasi-exactly solvable Hamiltonian is a polynomial of differential operators which form a finite-dimensional representation of a Lie algebra. This automatically guaranties the quasi-exact solvability of the Hamiltonian. In one-dimensional quantal problems the corresponding algebra is $\mathfrak{sl}(2, \mathbb{R})$. Later on such 1D systems were generalized to quasi-exactly solvable operators with invariant monomial subspaces [3] (for recent discussions see Refs. [4, 5]).

Recently it was revealed nontrivial relations between quasi-exactly solvable systems and nonlinear supersymmetry [6, 7, 8, 9] which is a generalization of the linear supersymmetric quantum mechanics [10, 11, 12] to the case of nonlinear (polynomial) superalgebras [13, 14]. The correspondence between the Lie-algebraic scheme based on the algebra $\mathfrak{sl}(2, \mathbb{R})$ and supersymmetric systems with nonlinear superalgebras was shown in Ref. [8] while supersymmetric systems corresponding to the quasi-exactly solvable operators with invariant monomial subspaces were constructed in Ref. [15].

Intertwining relations underline the standard realization of the quantum-mechanical supersymmetry both linear and nonlinear. The only difference of the nonlinear case is that the corresponding intertwining differential operators are of higher orders. Intertwining relations are corner stone of various approaches such as shape invariance to constructing exactly solvable systems [12, 16]. In this paper we emphasize the role of intertwining relations in constructing quasi-exactly solvable operators and offer a generalization of the construction to the multidimensional case.

The paper is organized as follows. In the next section we discuss the relation of intertwining relations to quasi-exact solvability of one-dimensional operators. Besides we offer a natural algebraic generalization of intertwining relations to the multidimensional case. In the section 3 we consider definition and some aspects of the one-dimensional quasi-exactly solvable operators. In the section 4 the results of the Lie-algebraic scheme based on the algebra $\mathfrak{sl}(2, \mathbb{R})$ are reproduced in terms of the method based on the intertwining relations. In the section 5 we discuss application of the method to quasi-exactly solvable operators with invariant monomial spaces. An intimate relation of quasi-exactly solvable operators with two invariant monomial subspaces to a nonlinear parasupersymmetry of second order is outlined in the section 6. The section 7 is devoted to investigation of one-dimensional quantum-mechanical systems with linear and nonlinear supersymmetry from the viewpoint of quasi-exact solvability. In this section we construct a general quasi-exactly solvable operator with three-dimensional invariant subspace, which is related to a cubic supersymmetry. Besides, using the method based on intertwining relations two new examples of quasi-exactly solvable operators are constructed. The corresponding invariant subspaces do not belong to spaces with a monomial basis but can be considered as their one-parametric deformation. The conclusion discusses and summarizes the obtained results.

2 Intertwining relations and quasi-exact solvability

In a general sense a differential operator is called quasi-exactly solvable if it has a finite-dimensional invariant subspace. If this subspace is known explicitly the corresponding part of spectral problem can be solved algebraically.

Let us consider the following intertwining relations

$$AH_0 = H_1 A, \qquad A^\dagger H_1 = H_0 A^\dagger, \tag{1}$$

where H_0, H_1 and A are some operators on a Hilbert space $\mathcal{H}$ and † denotes an involution, e.g. a hermitian conjugation. As is well known the intertwining operator A relates the spectra of the operators H_1 and H_0:

$$\sigma(H_0) \xrightarrow{A} \sigma(H_1), \qquad \text{i.e.} \qquad \tilde{\psi}_E = A\psi_E$$

for $\psi_E \notin \mathsf{Ker}\, A = \{\psi \in \mathcal{H} \mid A\psi = 0\}$, where E stands for an eigenvalue. Besides, from the intertwining relations (1) it follows that if $\mathsf{Ker}\, A$ is not empty then it is a nontrivial invariant subspace of the operator H_0:

$$\mathsf{Ker}\, A \xrightarrow{H_0} \mathsf{Ker}\, A.$$

This property is very interesting from the viewpoint of quasi-exact solvability. Indeed, if $\mathsf{Ker}\, A$ is a finite-dimensional space then the operator H_0 is quasi-exactly solvable. Hence, for a given operator A with a finite number of zero-modes the intertwining relations (1) provide a constructive way to build corresponding quasi-exactly solvable operator H_0.

When the operators H_0 and H_1 are hermitian a superalgebra[1] can be associated with the intertwining relations (1). Indeed, introducing the pair of odd variables θ, $\theta^\dagger$, $\{\theta, \theta^\dagger\} = 1$, one can construct the supercharges and super-Hamiltonian:

$$Q = A\theta, \qquad Q^\dagger = A^\dagger\theta^\dagger, \qquad \hat{H} = H_0\theta\theta^\dagger + H_1\theta^\dagger\theta. \tag{2}$$

The supercharges commute with the super-Hamiltonian due to the intertwining relations (1). Operators $\hat{H}$, Q, $Q^\dagger$ generate a linear superalgebra [10, 11] when A is a linear differential operator. If A is a higher order differential operator the corresponding superalgebra is of nonlinear (polynomial) form [6, 7, 8, 9, 18].

The intertwining relations (1) are very useful for constructing one-dimensional quasi-exactly solvable operators since A being a one-dimensional differential operator always has a finite number of zero-modes. In multidimensional problems it is not the case and the scheme based on the intertwining relations should be modified in appropriate way. To this end we offer the following generalization of the intertwining relations (1):

$$A_i H_0 = \sum_{j=1}^{M} H_{ij} A_j, \qquad \text{for} \qquad i = 1, \ldots, M. \tag{3}$$

[1] The operators also can be pseudo-hermitian and in this case one can associate a pseudo-supersymmetry [17, 9] with them.

In this case if the operator-valued matrix H_{ij} is "irreducible", i.e. it is not of a block-diagonal form, and $H_{ii} \neq 0$ for all i, the space

$$\mathcal{F}(H_0) = \bigcap_{i=1}^{M} \mathsf{Ker}\, A_i \tag{4}$$

is invariant subspace of the operator H_0. In general, depending on the particular form of the operator-valued matrix H_{ij} there can be some other invariant subspaces of H_0. For example, consider following two cases:

$$A_1 H_0 = H_1 A_1, \qquad\qquad A_2 H_0 = H_2 A_2, \tag{5}$$

and

$$A_1 H_0 = H_{11} A_1, \qquad\qquad A_2 H_0 = H_{21} A_1 + H_{22} A_2. \tag{6}$$

In the former case if the intersection of $\mathsf{Ker}\, A_i$ is not empty there are three invariant subspaces of the operator H_0: $\mathsf{Ker}\, A_i \setminus (\mathsf{Ker}\, A_1 \cap \mathsf{Ker}\, A_2)$, $i = 1, 2$ and $\mathsf{Ker}\, A_1 \cap \mathsf{Ker}\, A_2$. In the latter there are two invariant subspaces: $\mathsf{Ker}\, A_1 \setminus (\mathsf{Ker}\, A_1 \cap \mathsf{Ker}\, A_2)$ and $\mathsf{Ker}\, A_1 \cap \mathsf{Ker}\, A_2$. More detailed analysis of the generalized intertwining relations (3) in the context of invariant subspaces of the operator H_0 is out of the scope of the present paper and will be given elsewhere. We just would like to add that although in the multidimensional case any space of zero-modes $\mathsf{Ker}\, A_i$ generally has infinite dimension the invariant subspace (4) can be finite-dimensional. Therefore the relations (3) can be used for constructing multidimensional quasi-exactly solvable systems.

3 Quasi-exactly solvable operators in one dimension

Let us consider the most general differential operator of second order:

$$H = -P(z)\partial_z^2 + Q(z)\partial_z + R(z), \tag{7}$$

where we assume that $z \in I = \{z \in \mathbb{R} \mid P(z) > 0\}$, $\partial_z = d/dz$, $P(z)$, $Q(z)$ and $R(z)$ are some real functions. Formally, the operator H is hermitian with respect to the following inner product:

$$\langle \phi,\, \psi \rangle_\rho = \int_I \phi(z)\psi(z)\rho(z)dz, \qquad\qquad \rho(z) = P(z)^{-1} \exp\left(-\int_c^z \frac{Q(y)}{P(y)}\, dy \right), \tag{8}$$

where $\phi(z)$ and $\psi(z)$ are real functions. The generalization to the supersymmetric case is straightforward. The wave functions become two component functions and the weight function $\rho(z)$ should be changed for a diagonal matrix defined from hermicity of the Hamiltonians H_0, H_1.

We require for the Hamiltonian (7) to be a covariant operator with respect to coordinate transformations. Therefore, under the change of variable $z \to x(z)$ the functions $P(z)$, $Q(z)$ and $R(z)$ transform as

$$\begin{aligned} \tilde{P}(x(z)) &= (\partial_z x(z))^2 P(z), \\ \tilde{Q}(x(z)) &= \partial_z x(z) Q(z) - \partial_z^2 x(z) P(z), \\ \tilde{R}(x(z)) &= R(z). \end{aligned} \tag{9}$$

When matrix elements of the Hamiltonian (7) are invariant under the coordinate transformations:

$$\langle \tilde{\psi}, \tilde{H}\tilde{\psi} \rangle_{\tilde{\rho}} = \langle \psi, H\psi \rangle_{\rho},$$

where we assume that the wave functions are normalized and transform as scalar fields, $\tilde{\psi}(x(z)) = \psi(z)$, while the weight function ρ transforms according to its definition (8).

Besides, the system (7), (8) has another local symmetry. The matrix elements are invariant under the so-called "gauge" transformations:

$$\tilde{\psi}(z) = e^{\alpha(z)}\psi(z), \qquad \tilde{H} = e^{\alpha(z)} H e^{-\alpha(z)}. \tag{10}$$

In terms of the coefficient functions of the Hamiltonian it reads as

$$\begin{aligned} \tilde{P}(z) &= P(z), \\ \tilde{Q}(z) &= Q(z) + 2\partial_z \alpha(z) P(z), \\ \tilde{R}(z) &= R(z) - \partial_z \alpha(z) Q(z) + \left(\partial_z^2 \alpha(z) - (\partial_z \alpha(z))^2\right) P(z). \end{aligned} \tag{11}$$

Form the definition (8) one can deduce the transformation for $\rho(z)$:

$$\tilde{\rho}(z) = \lambda e^{-2\alpha(z)} \rho(z).$$

The constant factor λ depends on the integration constant c in the definition of $\rho(z)$ (8) but it is irrelevant until we use normalized wave functions.

The Hamiltonian (7) and the inner product (8) can be represented in an explicitly covariant form with respect to the transformations (9) and (11). To this end, consider the following operator:

$$H = -\frac{1}{\sqrt{g}} \nabla_z (\sqrt{g} g^{zz} \nabla_z) + V(z), \tag{12}$$

where the metric is given by the quadratic form $ds^2 = g_{zz}(z) dz \otimes dz$, $g = \mathsf{det}\, \|g_{zz}\| = g_{zz}$, $g^{zz} g_{zz} = 1$ and $V(z)$ is a scalar function. $\nabla_z = \partial_z - A_z$, where A_z is component of the one-form $\mathcal{A} = A_z(z) dz$. The definitions of the metric and the one-form fix how their components change under the coordinate transformations while we assume that under the "gauge" transformations (10) the 1D vector potential transforms as an Abelian connection

$$\tilde{A}_z = A_z + \partial_z \alpha. \tag{13}$$

Therefore, the transformations (11) are true gauge transformations and ∇_z is the corresponding covariant derivative. Hence, the Hamiltonian (12) has an explicitly covariant form.

The appropriate invariant inner product has the form

$$\langle \phi,\, \psi \rangle = \int \phi(z)\psi(z)e^{-2\int^z \mathcal{A}}\sqrt{g}dz, \tag{14}$$

where the exponential factor provides the invariance of the inner product with respect to the gauge transformations (10), (13).

The link between the covariant formulation (12), (14) is given by the relations

$$P(z) = g^{zz}, \qquad Q(z) = 2g^{zz}\left(A_z + \Gamma^z_{zz}\right), \qquad R(z) = V(z) - A^z A_z + g^{zz}\nabla_z A_z,$$

where $\Gamma^z_{zz} = \partial_z \log\sqrt{g}$. So, there is a one-to-one correspondence between the coefficient functions of the Hamiltonian (7) and g_{zz}, A_z, $V(z)$.

Now let us discuss one-dimensional quasi-exactly solvable operators. If the operator (7) is quasi-exactly solvable it has a finite-dimensional invariant subspace, say,

$$\mathcal{F}(H) = \mathsf{span}\{f_1(z),\, f_2(z),\, \dots,\, f_n(z)\}, \tag{15}$$

where linearly independent real functions $f_i(z)$ form a basis of the subspace. One can think that the functions are ordered by number of their nodes. Hereinafter we will denote a finite-dimensional invariant subspace of an operator $\mathcal{O}$ as $\mathcal{F}(\mathcal{O})$.

In principle one can extend this definition and treat an operator as quasi-exactly solvable one if a part of its spectrum (such part can be infinite-dimensional) can be found algebraically. Here we discuss only quasi-exactly solvable operators with finite-dimensional invariant subspace.

Generally, by a change of the coordinate and the gauge transformation (10) one can reduce the invariant subspace (15) to the following form [3]:

$$\mathcal{F}(H) = \mathsf{span}\{1, z, f_3(z),\, f_4(z),\, \dots,\, f_n(z)\}, \tag{16}$$

From this form it is obvious that the invariant subspaces of dimensionality 1 and 2 are special cases and require a special treatment.

At the end of this section we would like to briefly discuss the general structure of quasi-exactly solvable operator. As was proved in Ref. [19] every linear or nonlinear quasi-exactly solvable operator with a quite general finite-dimensional invariant subspace (15) can be represented in the following form:

$$T_{qes}[\psi] = \sum_{i=1}^{n} f_i(z)\, F_i(L_1[\psi],\, \dots,\, L_n[\psi]) + \sum_{\alpha=1}^{l} A_\alpha[\psi]\, \mathcal{O}_\alpha[\psi], \tag{17}$$

where the $F_i \in \mathcal{C}^\omega(\mathbb{R}^n)$, $\mathcal{O}_\alpha$ are arbitrary operators, A_α are annihilating operators (annihilators) of the space (15), i.e. $A_\alpha[\psi] = 0$ for all $\psi \in \mathcal{F}(T_{qes})$, and L_i form a basis of the space of affine annihilators, i.e. $L_i[\psi] = \mathit{const} \in \mathbb{R}$ for all $\psi \in \mathcal{F}(T_{qes})$.

4 Intertwining relations and the $\mathfrak{sl}(2,\mathbb{R})$-based scheme

In this section we consider application of the intertwining relations (1) to the well-known case of quasi-exactly solvable operators corresponding to the Lie-algebraic scheme based on the algebra $\mathfrak{sl}(2,\mathbb{R})$.

The Hamiltonians H_0 and H_1 (1) are taken in the most general form

$$H_i = \, - \, P_i(z)\partial_z^2 + Q_i(z)\partial_z + R_i(z), \tag{18}$$

where $P_i(z)$, $Q_i(z)$ and $R_i(z)$ are arbitrary real functions. Here we treat the intertwining operator A as an annihilation operator of the corresponding invariant monomial subspace. Hence, the "bosonic" Hamiltonian H_0 is a candidate for a quasi-exactly solvable operator while the operator H_1 is its supersymmetric partner in the sense of the supersymmetric construction (1), (2).

In the case of the $\mathfrak{sl}(2,\mathbb{R})$-based scheme the invariant subspace can always be reduced to the form

$$\mathcal{F}_n = \mathsf{span}\big\{1, z, z^2, \ldots, z^{n-1}\big\}, \tag{19}$$

where $n \in \mathbb{N}$. It is the representation space of the algebra $\mathfrak{sl}(2,\mathbb{R})$ with generators given by the differential operators

$$J_n^+ = z^2\partial_z - (n-1)\, z, \qquad J_n^0 = z\partial_z - \frac{n-1}{2}, \qquad J_n^- = \partial_z. \tag{20}$$

Here we assumed that $n > 2$ since the cases $n = 1, 2$ should be analyzed separately.

In the standard approach any quasi-exactly solvable operator is taken as a polynomial of the generators (22). However we will obtain the general form of such an operator using the strategy based on the intertwining relations (1). This method is more general then the Lie-algebraic approach since, as we will see later, it is equally good for constructing any one-dimensional quasi-exactly solvable operators.

It is very easy to see that an annihilator of the monomial subspace (19) can be represented in the following simple form:

$$A_n = \partial_z^n. \tag{21}$$

If we find the operators H_0, H_1 and A_n which obey the intertwining relations (1) then H_0 is a quasi-exactly solvable operator with the finite-dimensional invariant subspace $\mathsf{Ker}\, A_n = \mathcal{F}_n$. The question on the normalizability of such states we leave aside in this paper.

Substituting the operators (18) and (21) into the intertwining relations (1) one gets a system of differential equations for the functions $P_i(z)$, $Q_i(z)$, and $R_i(z)$, $i = 1, 2$. This system can be divided into two parts. The fist one can be solved algebraically and allows to express the coefficient functions of H_1 in terms of those of H_0:

$$\begin{aligned}
&P_1(z) = P_0(z),\\
&Q_1(z) = Q_0(z) - nP_0'(z),\\
&R_1(z) = R_0(z) + nQ_0'(z) - \frac{n(n-1)}{2}P_0''(z).
\end{aligned}$$

The second part is an overdetermined system of differential equations for the functions $P_0(z)$, $Q_0(z)$ and $R_0(z)$:

$$m(m-1)P_0^{(n-m+2)}(z) - m(n-m+2)Q_0^{(n-m+1)}(z) - (n-m+1)(n-m+2)R_0^{(n-m)}(z) = 0,$$

where $m = 0, \ldots, n-1$. The equations with $m \in \{n-3,\, n-2,\, n-1\}$ are independent and give the following 9-parametric solution:

$$
\begin{aligned}
P_0(z) &= \sum_{m=0}^{4} p_m z^m, \\
Q_0(z) &= \sum_{m=0}^{2} q_m z^m + \frac{n-2}{2} P_0'(z), \\
R_0(z) &= \frac{(n-1)(n-2)}{6} P_0''(z) - \frac{n-1}{2} Q_0'(z) + v,
\end{aligned} \tag{22}
$$

where p_m, q_m, and $v \in \mathbb{R}$. This solution satisfies the complete overdetermined system of differential equations. The second order operator H_0 with the coefficient functions (22) has the same form as in the Lie-algebraic scheme based on the algebra $\mathfrak{sl}(2, \mathbb{R})$.

It is worth noting that the procedure of solving such a system of differential equations is so simple only in the case (19).

5 Generalized quasi-exactly solvable systems with invariant monomial subspaces

A generalization of one-dimensional quasi-exactly solvable systems corresponding to the $\mathfrak{sl}(2, \mathbb{R})$-based approach was investigated for the cases of invariant subspaces with a monomial basis [3] (for more recent discussion, see Refs. [4, 5]). In this section we demonstrate that the approach to quasi-exact solvability based on the intertwining relations (1) can be applied to such quasi-exactly solvable systems as well.

Now let us discuss how one can deal with a general monomial invariant subspace using the intertwining relations (1). An n-dimensional space with monomial basis functions can be represented as

$$
\mathcal{F} = \mathsf{span}\big\{ z^{m_1},\, z^{m_2}, \ldots,\, z^{m_n} \big\}. \tag{23}
$$

One can consider the parameters m_i as natural numbers but for the method based on the intertwining relations it does not matter and m_i can be treated as real numbers.

The annihilator of the space (23) can be taken in the form

$$
A_n = \prod_{i=1}^{n} (z\partial_z - m_i). \tag{24}
$$

The intertwining relations (1) with the intertwining operator A_n lead to the overdetermined system of differential equations on coefficient functions of the operators H_0 and H_1. The particular solution of the system depends on the parameters m_i and order of the operators H_0 and H_1. A solution always exists since H_0 taken as a polynomial of the operator $z\partial_z$ leaves the subspace (23) invariant and the form of coefficient functions of the operator H_1 can be found from the intertwining relations algebraically. We are interested in nontrivial solutions for the differential operators H_0 and H_1 of second order.

In terms of the new variable $\xi = \log z$ one has $z\partial_z = \partial_\xi$ and the annihilator (24) becomes a differential operator with constant coefficients. As a consequence, the intertwining relations (1) lead to a system of differential equations with constant coefficients for the coefficient functions of the Hamiltonians H_0 and H_1. A system of this kind can be solved by a purely algebraic procedure. Therefore the resulting Hamiltonian H_0 is a quasi-exactly solvable operator with the invariant subspace (23).

Here we will not elaborate this procedure in any detail. Linear differential operators with finite-dimensional invariant subspaces of monomials were investigated in Ref. [3][2], where a classification of second order operators with invariant subspaces of the form (23) was given.

We give here the classification of such finite-dimensional invariant subspaces with basis of monomial functions without any explicit form of the corresponding operators. The form of the operators can be found in Refs. [3, 4, 5]. Invariant subspaces of monomials for nontrivial second order differential operators[3] can be divided into two classes. The fist one comprises the following spaces with a monomial basis of dimension 3 and 4:

$$\mathcal{F}_3 = \mathsf{span}\{1,\, z,\, z^m\}, \qquad\qquad \mathcal{F}_4 = \mathsf{span}\{1,\, z,\, z^m,\, z^{2m}\}.$$

The parameter m can be treated as real number. The second class is represented by series of spaces of monomials. The first series is that corresponding to the Lie-algebraic approach (19). The next series is

$$\mathcal{F} = \mathsf{span}\{1,\, z,\, z^2,\, \ldots,\, z^{n-2},\, z^n\}, \tag{25}$$

where $n \in \mathbb{Z}$. The last series is a direct sum of two subspaces:

$$\mathcal{F} = \mathsf{span}\{1,\, z,\, z^2,\, \ldots,\, z^{n-1}\} \oplus \mathsf{span}\{z^m,\, z^{m+1},\, z^{m+2},\, \ldots,\, z^{m+k-1}\} = \mathcal{F}_n \oplus z^m \mathcal{F}_k, \tag{26}$$

where $n,\, k \in \mathbb{N}$ and $m \in \mathbb{R}$.

Up to a change of variable and gauge transformations these spaces exhaust the list of invariant monomial subspaces of quasi-exactly solvable differential operators of second order.

All these cases can be reproduced in terms of the usual intertwining relations (1). This method allows to automatically construct the associated supersymmetric systems with nonlinear superalgebras (details on the construction can be found in Ref. [15]). In the next section we will show that using the simplest generalized intertwining relations (5) a nonlinear parasupersymmetry can be naturally associated with quasi-exactly solvable operators invariant on the monomial spaces (26).

6 Systems with two invariant subspaces and nonlinear parasupersymmetry

In quasi-exactly solvable systems with two invariant subspaces (26) one can associate an annihilating operator with each subspace. Consider the following two operators:

$$A_1 = \partial_z^n, \qquad\qquad A_2 = \partial_z^k z^{-m}. \tag{27}$$

[2]More recent discussion on the subject can be found in Refs. [4, 5].

[3]Here the trivial operator is a polynomial in $z\partial_z$.

Their spaces of zero-modes are

$$\mathsf{Ker}\, A_1 = \mathcal{F}_n, \qquad\qquad \mathsf{Ker}\, A_2 = z^m \mathcal{F}_k. \tag{28}$$

If the Hamiltonian H_0 is a second order differential operator and obeys the generalized intertwining relations (5) for some operators H_1 and H_2 then it is a quasi-exactly solvable operator with the invariant subspace (26)

$$\mathcal{F}(H) = \mathsf{Ker}\, A_1 \oplus \mathsf{Ker}\, A_2 = \mathcal{F}_n \oplus z^m \mathcal{F}_k.$$

From the first relation in (5) it follows that the Hamiltonian H_0 is a quasi-exactly solvable operator corresponding to the $\mathfrak{sl}(2,\mathbb{R})$-based scheme. The second relation from (5) can be considered as a constraint on this quasi-exactly solvable operator. Thus, without explicit calculations we came to the same conclusion as made in Ref. [4]. We can even say something more about this system. It is known that the generalized intertwining relations (5) underlay a parasupersymmetric system [20]. Indeed, from the relations (5) one concludes

$$A_i^\dagger A_i = P_i(H_0), \qquad\qquad A_i A_i^\dagger = P_i(H_i),$$

where $i = 1, 2$, $P_1(.)$ is a polynomial of order n and $P_2(.)$ is a polynomial of order k. Here the hermitian conjugation is understood with respect to the quite straightforward generalization of the inner product (8) to the case of 3-component wave functions and ρ changed for a 3×3 diagonal matrix defined from hermicity of the Hamiltonians H_0, H_1, and H_2. If one introduces the new operators

$$a_1 = A_1, \qquad\qquad a_2 = A_2^\dagger \tag{29}$$

the spectra of the Hamiltonians H_0, H_1 and H_2 are related as follows

$$\sigma(H_1) \xrightarrow{a_1^\dagger} \sigma(H_0) \xrightarrow{a_2^\dagger} \sigma(H_2), \qquad\qquad \sigma(H_1) \xleftarrow{a_1} \sigma(H_0) \xleftarrow{a_2} \sigma(H_2).$$

Let us introduce the following matrix operators

$$\hat{H} = \begin{pmatrix} H_2 & 0 & 0 \\ 0 & H_0 & 0 \\ 0 & 0 & H_1 \end{pmatrix}, \qquad\qquad Q = \begin{pmatrix} 0 & 0 & 0 \\ a_2 & 0 & 0 \\ 0 & a_1 & 0 \end{pmatrix}. \tag{30}$$

These operators form the nonlinear algebra given by

$$[\hat{H},\, Q] = 0, \qquad Q^2 Q^\dagger + Q Q^\dagger Q + Q^\dagger Q^2 = Q\big(P_1(\hat{H}) + P_2(\hat{H})\big), \qquad Q^3 = 0, \tag{31}$$

and the hermitian conjugated relations. It is easy to see that this algebra is a nonlinear version of the linear parasupersymmetry of order $p = 2$ [20]. This allows to conclude that the one-dimensional quasi-exactly solvable systems with two invariant subspaces are naturally related not only to the nonlinear supersymmetry but to the nonlinear parasupersymmetry as well.

7 Supersymmetric quantum mechanics from viewpoint of quasi-exact solvability

As it was discussed in Section 2 the intertwining relations (1) underline the general construction of one-dimensional supersymmetric quantum mechanics. If the operator A is a linear operator one arrives to the usual supersymmetric quantum mechanics [10, 11] while the construction with A to be a higher order differential operator corresponds to a nonlinear generalization of the supersymmetry with a polynomial superalgebra [6, 7, 8, 9, 18]. Owning to the discussed above correspondence between the intertwining relations (1) and quasi-exact solvability of one-dimensional systems we can consider the supersymmetric quantum mechanics in this context.

In a general quantum mechanical system with a linear supersymmetry the "bosonic" Hamiltonian can be treated as a quasi-exactly solvable operator with a one-dimensional invariant subspace. Since the sequence of eigenfunctions of the Hamiltonian can always be reduced to the form (16) without loss of generality we can take the invariant subspace in the form $\mathcal{F}(H) = \{1\}$. It is easy to see that if the second order differential operator is invariant on this space then it has the following form:

$$H = \, - P(z)\partial_z^2 + Q(z)\partial_z + const. \tag{32}$$

After changing the variable

$$x = \int \frac{dz}{\sqrt{P(z)}}$$

and introducing the notation

$$W(x) = \left. \frac{P'(z) + 2Q(z)}{4\sqrt{P(z)}} \right|_{z=z(x)}$$

the Hamiltonian (32) acquires the standard form

$$H = \, - \partial_x^2 + W(x)^2 - W'(x) + const.$$

One can note that while the operator (32) depends on two arbitrary functions the resulting operator of the Schrödinger form counts on just one function, the superpotential.

It is interesting to note that in this case the set of annihilating operators is infinite-dimensional. Its basis can be chosen in the form $L_n = z^{n+1}\partial_z$, $n \in \mathbb{Z}$, hence, these operators form the Virasoro algebra without central extension. The operator (32) can be treated as a combination of quadratic and linear terms in the generators of the Virasoro algebra. One can conclude that this fact is responsible for the presence of the functional arbitrariness in the supersymmetric Hamiltonian.

Now consider the general case of a second order quasi-exactly solvable operator with two-dimensional invariant subspace. Without lost of generality one can deal with the following subspace

$$\mathcal{F}(H) = \mathsf{span}\,\{1,\, z\}\,. \tag{33}$$

The corresponding quasi-exactly solvable Hamiltonian can be written as

$$H = \, - \, P(z) \partial_z^2 + \sum_{a=1}^{3} C_a J^a_{1/2} + const, \tag{34}$$

where C_a are constant and $J^a_{1/2}$ are linear differential operators (22) forming a representation of the algebra $\mathfrak{sl}(2,\, \mathbb{R})$ on the two-dimensional space (33).

The structure of the operator (34) with the two-dimensional invariant subspace is different as against the previous case. The second term in (34) is a linear combination of affine annihilators of the invariant subspace while the first one can be treated as a linear combination of annihilators for that space.

The annihilators of the space (33) are any differential operators of order 2 and higher without linear in derivative and multiplicative terms. Obviously, they form an infinite-dimensional set. The basis of this set can be taken in the form

$$L_n = z^{n+2} \partial_z^2, \qquad\qquad n \in \mathbb{Z}. \tag{35}$$

These operators together with the affine annihilators $J^a_{1/2}$ generate the following nonlinear infinite-dimensional algebra:

$$\big[L_m, L_n \big] = (m-n) L_{m+n} \circ J^0_{1/2}, \quad \big[J^0_{1/2}, L_m \big] = m L_m, \quad \big[J^{\mp}_{1/2}, L_m \big] = (m \mp 2) L_{m \pm 1}, \tag{36}$$

where $\circ$ stands for anticommutator. Therefore, one can conclude that the annihilators (35) together with the affine annihilators form the nonlinear (quadratic) generalization of the Virasoro algebra.

Using the usual intertwining relations (1) with the annihilator $A = \partial_z^2$ one can reproduce the form of the Hamiltonian (34) and construct its superpartner. That would complete the construction of the supersymmetric system with a quadratic superalgebra [13].

Now let us investigate the case of a three-dimensional invariant subspace, which can always be reduced to the following form:

$$\mathcal{F}(H) = \mathsf{span}\left\{ 1,\, z,\, f(z) \right\}. \tag{37}$$

Because of the presence of the arbitrary real function $f(z)$ this case looks different in comparison with the two previous cases. As we shall see it is also different from the viewpoint of quasi-exact solvability of second order operators.

Using the usual intertwining relations (1) with the annihilator of the subspace (37)

$$A = \partial_z^3 - \big(\log f''(x) \big)' \partial_z^2 \tag{38}$$

one can derive the form of corresponding Hamiltonian:

$$H = \sum_{i,j=1}^{3} f_i(z) \, a_{ij} \, K_j, \tag{39}$$

where the functions

$$f_1(z) = 1, \qquad\qquad f_2(z) = z, \qquad\qquad f_3(z) = f(z) \tag{40}$$

form the basis of the invariant subspace (37) and a_{ij} are real numbers. The operators K_i form a basis of affine annihilators of the subspace (37):

$$K_1 = \mathbb{1} - zK_2 - f(z)K_3, \qquad K_2 = \partial_z - \frac{f'(z)}{f''(z)}\partial_z^2, \qquad K_3 = f''(z)^{-1}\partial_z^2.$$

They act on the basis (40) as follows:

$$K_i[f_j] = \delta_{ij}.$$

From the viewpoint of the general structure (17) of quasi-exactly solvable operators the Hamiltonian (39) is just a linear combination of affine annihilators. Any annihilator of the invariant subspace (37) is a differential operator of order not less then 3. Therefore, the Hamiltonian cannot contain any of them since we are restricted by the class of second order differential operators.

The form of the supersymmetric partner of the Hamiltonian (39) can be derived from the corresponding intertwining relations. We do not need to know it to obtain the form of the superalgebra they generate together with the supercharges (see (2)). The conservation of supercharges follows from the intertwining relations. The anticommutator of supercharges is proportional to a cubic polynomial in the super-Hamiltonian:

$$\left\{Q,\, Q^\dagger\right\} = \mathsf{Det}\,(E\mathbb{1} - A)\big|_{E\to\hat{H}} = \hat{H}^3 + \ldots,$$

where elements of the matrix $A = \|a_{ij}\|$ are defined in (39). This follows form the fact that the finite-dimensional invariant subspace (37) of the "bosonic" Hamiltonian (39) corresponds to the zero-mode space of the supercharge Q.

Construction of the cubic supersymmetry with Hamiltonians in Schrödinger form was presented in Ref. [21]. Other approaches to general quasi-exactly solvable operators with three states can be found in Ref. [22].

Now we would like to discuss the general case of one-dimensional quasi-exactly solvable differential operators of second order. If the operator (7) has a n-dimensional invariant subspace (15) its coefficient functions and the basis functions $f_i(z)$ obey the system of differential equations

$$-P(z)f_i''(z) + Q(z)f_i'(z) + R(z)f_i(z) = \sum_{j=1}^{n} C_{ij}f_j(z), \tag{41}$$

$i = 1, \ldots, n$, for some real constants C_{ij}. If $n \le 3$ the equations (41) can be treated as a system of algebraic equations for the variables $P(z)$, $Q(z)$, and $R(z)$. In this manner the Hamiltonians (32), (34), and (39) can be reproduced. Beginning with $n = 4$ the situation changes drastically. Indeed, in the case $n = 4$ the system allows not only to fix the coefficient functions of the Hamiltonian but it also leads to a nonlinear differential equation for the functions $f_i(z)$.[4]

Besides, from the system (41) one can easily infer that in the cases $n = 1, 2$ the functional arbitrariness of the Hamiltonians is related to the degeneracy of the corresponding algebraic system. The system (41) also allows to conclude that the nature of the functional

[4] Or for the functions $f_3(z)$ and $f_4(z)$ in the case of the reduced invariant subspace (16).

arbitrariness in the Hamiltonian (39) is different as against the cases (32) and (34). For $n = 1$ and $n = 2$ the functional arbitrariness is related to the degeneracy of the system of algebraic equations on the coefficient functions while for $n = 3$ it is related to the presence of the arbitrary function in the basis (16).

The Hamiltonian restricted to the invariant subspace is represented by the matrix C_{ij}. Although it is real valued but nonsymmetric matrix, hence, its eigenvalues can be complex numbers. Therefore, if C_{ij} is nonsymmetric matrix one has to assume that the functions forming the basic of the subspace do not have finite norm with respect to the inner product (8) since the Hamiltonian has to be a hermitian operator. General structure of the matrix C_{ij} was studied in Ref. [18] in the context of polynomial superalgebras.

In the general context the most nontrivial problem is to find and classify all functional finite-dimensional spaces which can serve as invariant subspaces of differential operators of the general form (7) and cannot be reduced to the monomial form using changes of the variable and gauge transformations. Such spaces do exist. For example, consider the following one-parametric deformation of the series (25):

$$\mathcal{F} = \mathsf{span}\Big\{1,\, z,\, z^2,\, \ldots,\, z^{n-3},\, z^{n-2} + \frac{\alpha}{n-1} z^{n-1},\, z^n\Big\} \tag{42}$$

with $n \in \mathbb{N}$ and $\alpha \in \mathbb{R}$, which cannot be reduced to the monomial form for $\alpha \neq 0$. The corresponding annihilator of the subspace can be represented in the form

$$A_n = \big((\alpha z + 2)\partial_z - \alpha\big)(z\partial_z - 2)\partial_z^{n-2}.$$

Application of the method of intertwining relations leads to a 6-parametric quasi-exactly solvable Hamiltonian which is particular case of the $\mathfrak{sl}(2,\mathbb{R})$-based Hamiltonian with $n \to n-2$ and the following constraints on the parameters:

$$p_4 = 0, \quad q_0 = \frac{n+2}{2} p_1 - \alpha p_0, \quad q_1 = -\frac{2}{\alpha}(n-1)\, p_3 + n p_2 + \alpha p_1 - \alpha^2 p_0, \quad q_2 = \frac{n+4}{2} p_3.$$

As another example of invariant subspaces of monomials consider the following space:

$$\mathcal{F} = \mathsf{span}\big\{1,\, z,\, z^3,\, z^4 + 6\alpha z^2\big\}.$$

Using the method of intertwining relations one calculates the 7-parametric quasi-exactly solvable Hamiltonian given by the coefficient functions

$$\begin{aligned}
P(z) &= p_5 z^5 + p_3 z^3 + p_2 z^2 + p_1 z + p_0,\\
Q(z) &= 6 p_5 z^4 + 4\,(p_3 + 4\alpha p_5)\, z^2 + \big(5p_2 - \alpha^{-1} p_0\big)\, z - 8\alpha^2 p_5 + 4\alpha p_3 + 2p_1,\\
R(z) &= -12 p_5 z^3 - 4\,(13\alpha p_5 + p_3)\, z + r_0.
\end{aligned}$$

More detailed discussion of quasi-exactly solvable operators with non-monomial invariant subspaces will be presented elsewhere.

8 Conclusion

In this paper we emphasized the method based on the intertwining relations as a general approach to constructing one-dimensional quasi-exactly solvable operators. We demonstrated the universality of this method, which is based on the algebraic nature of the intertwining relations. The discussed examples cover all quasi-exactly solvable operators with invariant monomial subspaces. It can be equally applied to quasi-exactly solvable operators with non-monomial invariant subspaces. Using the method we constructed two examples of such operators with invariant subspaces which cannot be reduced to a monomial space.

The intertwining relations themselves can be applied to the one-dimensional case only. We offered natural algebraic generalization that can be applied to the multidimensional case as well. We showed that the simplest form of the generalization allows to relate the quasi-exactly solvable systems with two invariant monomial subspaces to a nonlinear parasupersymmetry of second order. The application of the generalized intertwining relations to constructing quasi-exactly solvable operators in the multidimensional case will be presented in future publications.

Besides we investigated quantum-mechanical systems both with linear and nonlinear supersymmetry from the standpoint of quasi-exact solvability. We demonstrated that in the cases of linear and quadratic supersymmetry the functional arbitrariness of the Hamiltonian is related to an infinite-dimensional algebra of annihilators of corresponding invariant subspaces. It is the Virasoro algebra without central extension in the first case and a quadratic deformation of the Virasoro algebra in the second. Also we constructed a general quantum-mechanical system associated with a cubic supersymmetry and argued that quantum-mechanical systems with nonlinear supersymmetry of order more then three are not quasi-exactly solvable in general.

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